Chapter 13

Conditional Probability

A Question to Start

A rapid antigen test is used to screen for COVID-19.
Based on a large validation study (117,372 samples):

- 5% of the population currently has COVID.
- If you *have* COVID, the test gives a positive result 73% of the time.
- If you *don't* have COVID, the test incorrectly gives a positive result 0.7% of the time.

You take the test and receive a **positive** result.

What is the probability you actually have COVID?

The Answer Depends on How Common the Disease Is

The same test gives very different answers depending on COVID prevalence:

During a wave (5% prevalence)	
Between waves (0.5% prevalence)	

The test itself does not change. The accuracy does not change. But how common the disease is changes the answer dramatically.

Human intuition struggles with conditional probability when base rates are involved.

Why Does the Base Rate Matter?

When prevalence is low and you test positive, there are two possible explanations:

Rare

True positive
You have COVID and
the test correctly detected it.

Common

False positive
You do not have COVID but
the test incorrectly flagged you.

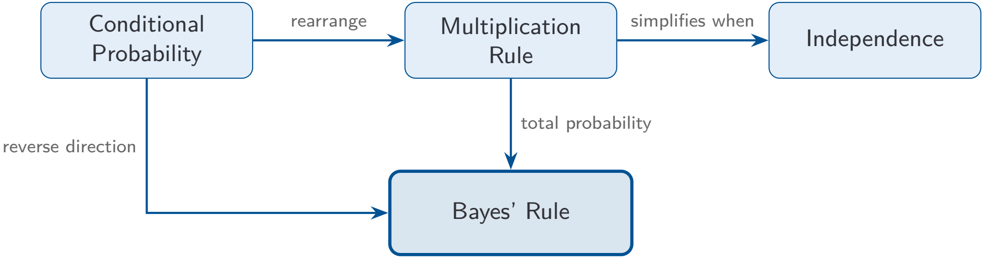
When a disease is rare, the second explanation (false positive) is more likely than the first.

Intended Learning Outcomes

- Define and compute conditional probabilities
- Apply the multiplication rule
- Use tree diagrams for multi-stage problems
- Define and test for independence
- Use the Law of Total Probability
- Apply Bayes' Rule to reverse conditionals
- Distinguish independence from mutual exclusivity

Chapter 13: The Big Picture

How does learning new information change probabilities?



PART 1

Conditional Probability

How does knowing B occurred change the probability of A?

Updating Probability with New Information

How does learning new information change what we believe?

- You rolled an even number. What is the probability it was a 2?
- A patient tests positive for a disease. What is the probability they actually have it?
- How can we use conditional probability to obtain estimates for the prevalence of a stigmatized behaviour, like infidelity?

Terminology

Joint and Marginal Probabilities

$P(A \cap B)$: **joint probability**, the probability both events occur.

$P(A)$, $P(B)$ alone: **marginal probabilities**, the probability of a single event.

Conditioning on B means we zoom in on outcomes where B occurred and ask what fraction are also in A.

Conditional Probability: Definition

Conditional Probability

The **conditional probability** of event A given event B has occurred is:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

provided that $P(B) > 0$.

- The bar "$|$" is read "given"
- Denominator $P(B)$ restricts to outcomes where B occurred
- Conditioning "zooms in" on the subset of outcomes where B happened

Conditional Probability with a Die

Example 13.1

Roll a fair six-sided die. Let $A =$ "roll a 2" and $B =$ "roll is even."

Calculate: The conditional probabilities below.

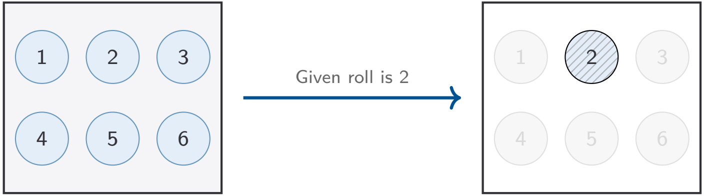
(a) Find $P(A \mid B)$:

(b) Find $P(B \mid A)$:

Visualizing conditional probabilities

$P(A \mid B)$

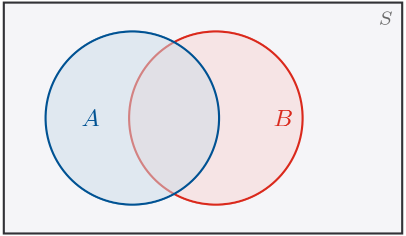
$P(B \mid A)$



Visualizing Conditional Probability

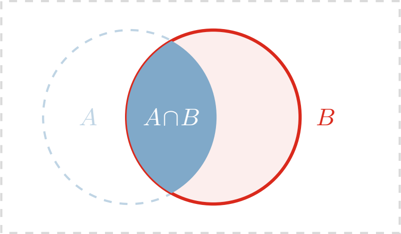
Example 13.2

Marginal: $P(A)$



$$P(A) = \frac{|A|}{|S|}$$

Conditional: $P(A \mid B)$



$$P(A \mid B) = \frac{|A \cap B|}{|B|}$$

Principle: Conditioning on B replaces S with B. What fraction of B lies in A?

A Subtle Difference in Conditioning

Example 13.3

Context: Pavlov has two dogs. Each dog is equally likely to be male (M) or female (F), independently. The sample space is

$$\{MM, MF, FM, FF\},$$

each with probability $\frac{1}{4}$.

Which probability is larger?

(a) $P(\text{both male} \mid \text{first dog is male})$

(b) $P(\text{both male} \mid \text{at least one dog is male})$

Pavlov's Dogs: Calculating the Probabilities

(a) $P(\text{both male} \mid \text{first is male})$

(b) $P(\text{both male} \mid \text{at least one male})$

Why the Conditions Give Different Answers

Example 13.3: Visualization

PART 2

The Multiplication Rule

Finding joint probabilities from conditional ones

From Conditional to Joint Probability

If we know $P(A \mid B)$, can we find $P(A \text{ and } B)$?

Starting from the definition and rearranging:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Multiplication Rule

For any events A and B:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

$P(A \text{ and } B)$ equals $P(B)$ times the probability that A also occurs given B.

Applying the Multiplication Rule

Example 13.4

Context: A university health survey of students aged 18–22 found that social media use before bed affects sleep quality.

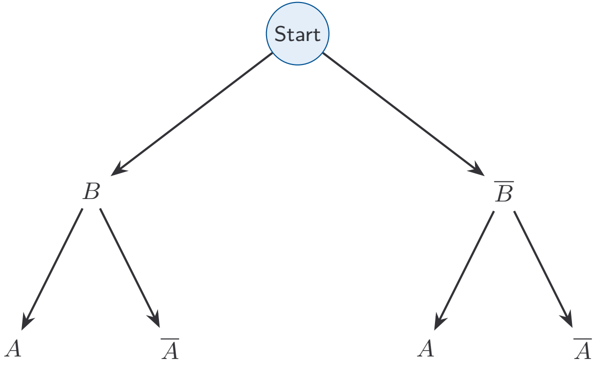
- 60% of students spend more than 2 hours on social media before bed.
- Of heavy users, 70% report poor sleep quality.
- Of light users ($\leq$ 2 hours), 25% report poor sleep quality.

Find: $P(\text{heavy user and poor sleep})$

Tree Diagrams: A Visual Tool for Multi-Stage Problems

Tree Diagram

A **tree diagram** displays all possible outcomes of a sequence of events, with each branch showing a conditional probability.



Principle: To find the probability of a path through the tree, **multiply** the probabilities along the branches.

Tree Diagram for Drawing Without Replacement

Example 13.5

Context: A box has 5 red and 3 blue marbles. You draw two marbles one at a time without putting the first one back.
Calculate the following probabilities

(a) $P(\text{both red})$

(b) $P(\text{2nd blue} \mid \text{1st red})$

(c) $P(\text{exactly one red})$

Choosing the Order of Branches

The order of branches does not affect the final probabilities, but it can make the computation easier.

1. Start with the event whose unconditional probability is known, then branch to conditional probabilities.
2. If the events happen in sequence (e.g. drawing marbles one at a time), order the branches in the same sequence.

> **Why it works:** Think of the first branch as "what do we already know?" and the following branches as "what happens next, given what we know?"

COVID Testing: Applying the Multiplication Rule

Example 13.6

Context: Recall the rapid antigen test from the start of this chapter:

- 5% of the population currently has COVID ($P(D) = 0.05$).
- If a person has COVID, the test is positive 73% of the time ($P(+ \mid D) = 0.73$).

Find: $P(\text{has COVID and tests positive})$

This gives one "pathway" to a positive test. But there is another: people who don't have COVID but test positive anyway. We will learn how to combine both pathways in Part 4.

PART 3

Independence

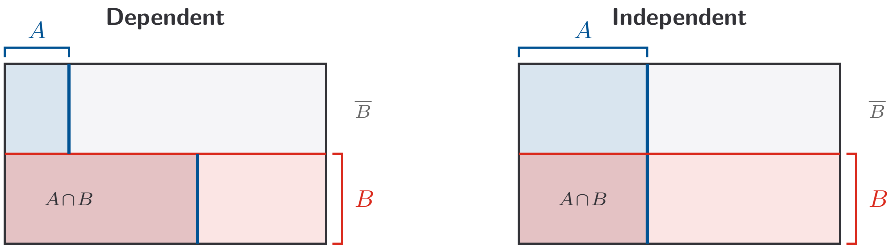
When does knowing B tell us nothing about A?

Independent Events: Definition

Independence

Two events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B)$$



$P(A \mid B) \neq P(A)$

$P(A \mid B) = P(A)$

Multiplication Rule for Independent Events

Example 13.7

Context: Roll a fair six-sided die. Let $A =$ "roll a 2" and $B =$ "roll is even."

Determine if A and B independent?

An Equivalent Condition for Independence

If A and B are **independent**, and $P(B) > 0$, then

$$P(A \mid B) = P(A)$$

> **Principle:** Knowing that B occurred does not change the probability of A.
>
> This is often the most intuitive way to think about independence.

Testing for Independence

Example 13.8: Setup

Context: Does binge-watching Netflix the night before class make students more likely to miss class? A survey of 200 university students found:

- 80 binged Netflix ($>$3 hours the night before); of these, 36 missed their next class.
- 120 did not binge; of these, 14 missed their next class.

Are binge-watching (N) and missing class (M) independent?

Testing for Independence

Example 13.8: Calculation

Context: 200 students. Binged Netflix: 80 (36 missed class). Did not binge: 120 (14 missed class).

(a) $P(M)$

(b) $P(M \mid N)$

(c) Are they independent?

PART 4

Law of Total Probability & Bayes' Rule

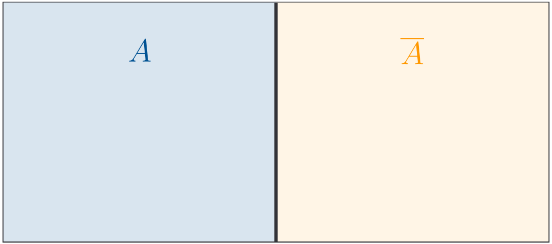
Reversing the direction of conditional probability

Partitioning the Sample Space

Partition

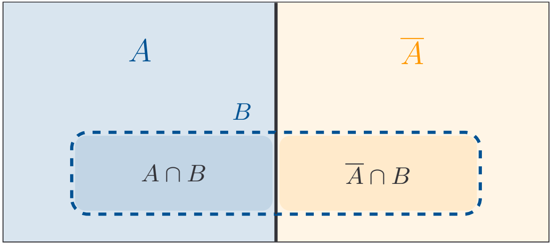
Events $A_1, A_2, \ldots, A_k$ form a **partition** of the sample space S if:

1. They are **mutually exclusive**: no two can occur at the same time.
2. They are **exhaustive**: together they cover all of S.



Splitting B Across the Simplest Partition

Event B split by the partition A and $\overline{A}$



Every outcome in B is either in A or in $\overline{A}$, so:

$$P(B) \;=\; P(A \cap B) \;+\; P(\overline{A} \cap B) \;=\; P(B \mid A)\, P(A) \;+\; P(B \mid \overline{A})\, P(\overline{A})$$

Returning to COVID-19 Testing

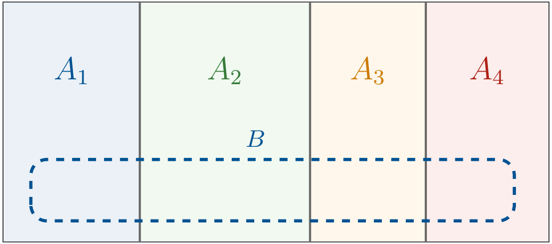
Example 13.9

Recall the rapid antigen test data from the start of this chapter:

- 5% of the population currently has COVID ($P(D) = 0.05$).
- If a person has COVID, the test is positive 73% of the time ($P(+ \mid D) = 0.73$).
- If a person does not have COVID, the test is positive 0.7% of the time ($P(+ \mid \overline{D}) = 0.007$).

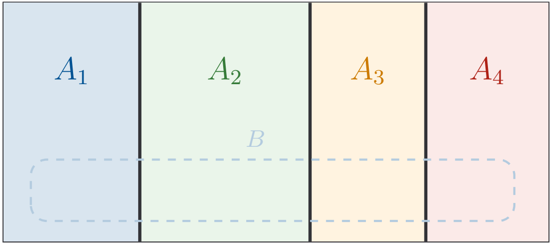
Find $P(+)$, the overall probability of testing positive.

A More Complex Partition



The Partition: Mutually Exclusive and Exhaustive

A_1, A_2, A_3, A_4 partition the sample space S



Principle: The A_i are mutually exclusive (no overlaps) and exhaustive (no gaps).

Every outcome in S belongs to exactly one A_i.

Chip Quality Control

Example 13.10: Setup

Context: A factory produces semiconductor chips on three production lines. The table shows each line's share of total output and its defect rate.

Line	Share of Output	Defect Rate
Line A	50%	2%
Line B	30%	4%
Line C	20%	5%

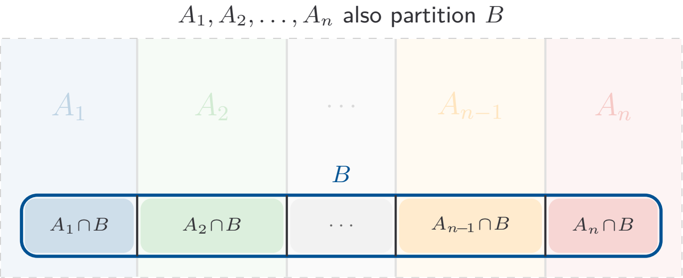
A chip is selected at random. What is the probability it is defective?

Chip Quality Control

Example 13.10: Calculation

Find: $P(D)$, the probability a randomly selected chip is defective.

The Partition Also Splits B



Principle: For *any* partition $A_1, A_2, \ldots, A_n$ of S, the same partition splits B into non-overlapping pieces

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

Law of Total Probability

Law of Total Probability

If $A_1, A_2, \ldots, A_k$ partition S, then

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)\, P(A_i)$$

for any event B.

Chip Quality Control

Example 13.10: Bayes Application

Context: Returning to the chip problem. A chip is selected at random and found to be defective. Which production line is it most likely from?

Line	**Share**	**Defect Rate**
Line A	50%	2%
Line B	30%	4%
Line C	20%	5%

The Problem: We Know the Wrong Direction

In many settings, the conditional probability we have access to often the reverse of the one we actually want.

- A medical test is 95% accurate for people with a disease.
  We know $P(+ \mid D)$. We need $P(D \mid +)$.

- A radar system correctly flags 90% of enemy aircraft.
  We know $P(\text{Flag} \mid \text{Enemy})$. We need $P(\text{Enemy} \mid \text{Flag})$.

- A factory inspection catches 98% of defective items.
  We know $P(\text{Caught} \mid \text{Defective})$. We need $P(\text{Defective} \mid \text{Caught})$.

Reversing Conditional Probabilities

Bayes' Rule lets us "flip" the direction of a conditional probability: from $P(B \mid A)$ to $P(A \mid B)$.

$$\text{ingredients}\left\{\begin{array}{c} P(B \mid A)\\ \text{known}\\[3ex] P(A)\\ \text{base rate}\end{array}\right. \quad + \quad \begin{array}{c}P(B)\\ \text{total prob.}\end{array} \quad \xrightarrow{\text{Bayes' Rule}} \quad \boxed{\begin{array}{c}P(A \mid B)\\ \text{what we need}\end{array}}$$

Bayes' Rule

Bayes' Rule

For events A and B with $P(A) > 0$, $P(B) > 0$

COVID-19 Rapid Antigen Tests

Example 13.11: Context

- **Base rate:** 5% of the population currently has COVID.
- **If a person has COVID:** The test correctly gives a positive result 73% of the time.
- **If a person does not have COVID:** The test incorrectly gives a positive result 0.7% of the time.
- We saw that the overall probability of testing positive is about 4.3% using the law of total probability.

Find: $P(\text{COVID} \mid +)$, the probability that a person who tests positive actually has COVID.

COVID-19 Rapid Antigen Tests

Example 13.11: Calculation

Recall that we know $P(+ \mid D) = 0.73$, $P(+ \mid \overline{D}) = 0.007$, $P(D) = 0.05$
Previously, we saw that $P(+) = 0.0432$ using the law of total probability.

Calculate: Use Bayes' Rule to find $P(\text{COVID} \mid +)$.

Battle of Britain: Radar Detection (1940)

Example 13.12: Context

During the Battle of Britain, radar operators at RAF Fighter Command had to decide whether approaching aircraft were enemy bombers or friendly planes returning from patrol.

A wrong decision meant either:

- Missing an enemy attack (catastrophic).
- Scrambling fighters against friendly aircraft (wasteful).

Battle of Britain: Radar Detection (1940)

Example 13.12: Setup

Context: Based on intelligence reports:

- 25% of approaching aircraft are enemy bombers.
- Radar correctly identifies 90% of enemy aircraft.
- Radar incorrectly flags 10% of friendly aircraft as enemy.

Find: Radar flags an aircraft as enemy. What is the probability it is actually hostile?

Battle of Britain: Radar Detection (1940)

Example 13.12: Calculation

Context: 25% enemy, 90% correct detection, 10% false positive on friendly.

Calculate: $P(\text{Enemy} \mid \text{Flagged})$

CHAPTER 13

Summary

Key concepts and formulas from this chapter

Chapter 13 Summary

Conditional Probability

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- Restricts sample space to B
- Direction matters.

Multiplication Rule

- $P(A \cap B) = P(B) \cdot P(A \mid B)$
- Extends to 3+ events
- Tree diagrams help visualize

Independence

- $P(A \mid B) = P(A)$
- If independent: $P(A \cap B) = P(A) \cdot P(B)$
- $\neq$ mutually exclusive.

Law of Total Probability

- $P(B) = \sum P(B \mid A_i)\, P(A_i)$
- Sums over all pathways to B

Bayes' Rule

- $P(A \mid B) = \frac{P(B|A) \cdot P(A)}{P(B)}$
- Reverses conditional direction

PRACTICE

Problems

Test your understanding of conditional probability, independence, and Bayes' Rule

Practice Problem 1: Conditional Probability

Context: A survey of 500 employees at a company:

	Full-Time	**Part-Time**	**Total**
Remote	120	80	200
In-Office	210	90	300
Total	330	170	500

(a) Find $P(\text{Remote})$.
(b) Find $P(\text{Remote} \mid \text{Part-Time})$.
(c) Find $P(\text{Full-Time} \mid \text{Remote})$.
(d) Are working remotely and being full-time independent? Justify.

Practice Problem 2: Multiplication Rule and Trees

Context: A bag contains 4 red, 3 green, and 2 blue marbles. You draw two marbles without replacement.

(a) Draw a tree diagram for the first two draws (group by colour).
(b) Find $P(\text{both marbles are red})$.
(c) Find $P(\text{second marble is green} \mid \text{first marble is red})$.
(d) Find $P(\text{at least one marble is blue})$.

Practice Problem 3: Independence

Context: At a gym, 60% of members use cardio equipment, 45% use weights, and 30% use both.

(a) Find $P(\text{weights} \mid \text{cardio})$.
(b) Find $P(\text{cardio} \mid \text{weights})$.
(c) Are using cardio and using weights independent events? Show your work.
(d) Are using cardio and using weights mutually exclusive? Explain.

Practice Problem 4: Bayes' Rule

Context: Three major U.S. airlines have the following market shares and on-time arrival rates (BTS/OAG, 2023):

- Delta: 35% of flights, 83% on-time
- United: 30% of flights, 79% on-time
- American: 35% of flights, 79% on-time

(a) If a random domestic flight is selected, what is the probability it arrives on time?
(b) If a flight is late, what is the probability it was a Delta flight?
(c) If a flight is late, which airline is it most likely from?

Practice Problem 5: Conceptual Questions

For each statement, determine if it is TRUE or FALSE. Justify your answer.

(a) If $P(A) = 0.3$ and $P(B) = 0.4$, and A and B are independent, then $P(A \text{ or } B) = 0.7$.

(b) **(Challenging)** If $P(A \mid B) = 0$, then A and B must be mutually exclusive.

(c) **(Challenging)** If A and B are independent, then A and $\overline{B}$ are also independent.

(d) **(Challenging)** $P(A \mid B) + P(\overline{A} \mid B) = 1$ for any events A, B with $P(B) > 0$.